# Application of Nonstandard Analysis to the Study of the Shock Structure in a Viscous Heat Conducting Fluid


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## Abstract


The study of the shock structure in a viscous heat conducting fluid is an old problem [Murduchow and Libby, J. Aero. Sc., Vol. 16, 1949, 674-704]. We study this problem from a novel mathematical point of view. A new class of generalized functions is defined where multiplication of any two functions is allowed with the usual properties. A Heaviside function in this class has the unit jump at $x = 0$ occurring on an infinitesimal interval $\varepsilon$ of the nonstandard analysis (NSA) in the halo of $x = 0$. This jump has a smooth microstructure over the infinitesimal interval $\varepsilon$. From this point of view, we have a new class of Heaviside functions, and their derivatives the Dirac delta functions, which are equivalent when viewed as continuous linear functionals over the test function space of Schwartz. However, they differ in their microstructures which in applications are determined from physics of the problem as shown in our presentation. We start by assuming that the jumps in fluid dynamic parameters pressure $p$, specific volume $v$, velocity $u_i$, etc., occur over the same infinitesimal interval $\varepsilon$. We emphasize that what we call a jump here has a smooth transition within an infinitesimal interval that to an observer on the real line looks like a classical jump obtained from the shock macrostructure. For each fluid dynamic parameter, say the pressure $p$, we write $p = p_1 + \Delta p L(x)$, where $p_1$ is the pressure on the upstream side of the

shock, $\Delta p$ is the jump across the shock and $L(x)$ is the Heaviside function associated with the transition of the pressure within the shock which is identically zero on the side of the shock where the pressure is $p_1$. The Heaviside functions associated with the jumps of the fluid dynamic parameters across the shock are assumed to have different microstructures. We are interested in determining the shock speed, the values of these jumps, and the microstructures of these Heaviside functions which collectively we call the shock structure. We study the shock structure for a fluid with constant viscosity and heat conductivity and Prandtl number $Pr = 3/4$. Let $E, H, K, L,$ and $N$ be the Heaviside functions associated with the jumps in entropy, specific volume, velocity, pressure and temperature, respectively, having possibly different microstructures. From conservation of mass equation, we show that $H = K$, i.e., they have the same microstructure. The shock speed is also obtained from this conservation law. From the momentum equation, we find one first order O.D.E relating $H$ and $L$. Using this result in the energy equation, we get a second order nonlinear O.D.E. in $H$ which can be solved numerically utilizing Mathematica 4. The jumps in parameters across the shock are also obtained in this process which are the same as those obtained by the classical method. From the knowledge of $H$, the Heaviside function $L$ is obtained. The other Heaviside functions are obtained from thermodynamic relations. The results show that the microstructures of $E, H, L,$ and $N$ are all different from each other. One of the most interesting results obtained is that while $H$ and L are monotonically increasing from zero to one within a shock, the Heaviside functions $E$ associated with the entropy jump has a positive peak of greater than one within the shock. The Heaviside function $N$ associated with temperature jump is monotonically increasing at low Mach numbers but develops a peak greater than one within the shock at high Mach numbers. These results have been obtained by other means before but it is satisfying to obtain them using NSA. We present numerical results of our work showing the dependence of the microstructure of the Heaviside functions on the upstream Mach number. The new generalized functions give us one more tool to study physical phenomena where two or more vastly different length or time scales are associated with the problem. The laws of physics will allow us to select the correct generalized function from the class of generalized functions with different microstructures that are equivalent when viewed as continuous linear functionals over the test function space of Schwartz.

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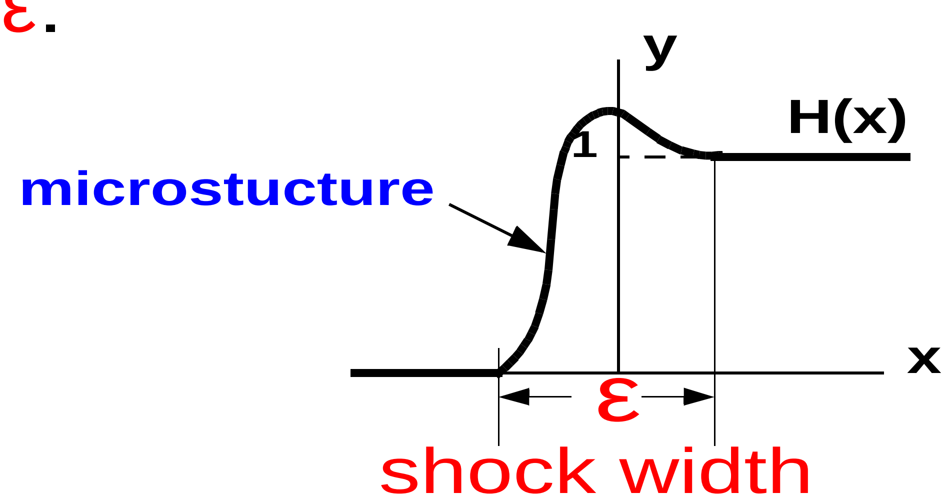
# Introduction

The study of shock structure is an old problem. We will study it using nonstandard analysis (NSA) assuming that the shock thickness is of an arbitrary infinitesimal length $\varepsilon$ of NSA. We will then introduce a macroscopically equivalent class of Heaviside functions (HF's) which differ in their microstructure over $\varepsilon$.



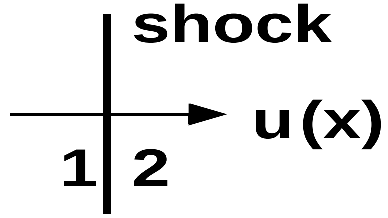
# Initial Step of the Analysis

**We consider a normal shock in a fluid with constant viscosity, heat conductivity and Prandtl number 3/4. Introduce five HF's E, H, K, L, and N which may differ in their microstructure. Let $\Delta$ denote the jump of a flow parameter across the shock. In streamwise direction *x*, we write**

$S(x) = S_1 + \Delta S E(x)$ **Entropy**, $v(x) = v_1 + \Delta v H(x)$ **Specific volume**

$u(x) = u_1 + \Delta u K(x)$ **Velocity**, $p(x) = p_1 + \Delta p L(x)$ **Pressure**

$T(x) = T_1 + \Delta T N(x)$ **Temperature**



# Assumptions and Properties

## Assumptions:

1. The microstructures of the HF of flow parameters are defined over the same infinitesimal interval $\varepsilon$

2. The variation of flow parameters over $\varepsilon$ is $C^{\infty}$

## Properties:

1. All the HF's are macroscopically identical to the usual HF
2. The derivatives of all orders of each HF are clearly defined over hyperreals
3. The product of a HF and a delta function is clearly defined over hyperreals. The product of two delta functions is also defined.
4. The new class of delta functions as derivatives of the new class of HF's behave as the well-known Dirac delta function on the space of test functions of schwartz distributions
5. Each HF in new class of HF's is discontinuous over hyperreals

# The Structure of the Shock in a Viscous Heat Conducting Fluid

## The Governing Equations

$$v_t + uv_x - vu_x = 0$$ **Mass Continuity**

$$u_t + uu_x + vp_x - \beta v u_{xx} = 0, \quad \beta = \frac{4\mu}{3}$$ **Momentum**

$$\alpha(pv)_t + \alpha u(pv)_x + pvu_x - \beta v u_x^2 = 0, \quad \alpha = \frac{1}{\gamma - 1}$$ **Energy**

**Assumptions: constant viscosity coefficient, nonheat conducting gas for the following analysis (constant viscosity coefficient and heat conductivity, Prandtl number = 3/4 for the figures)**

# Mathematical Analysis

**Let** $\xi = u - ct$, c **shock speed. All HF's are functions of** $\xi$**. Mass Continuity eq. gives:**

$$\frac{dH}{\Delta u(v_1 + \Delta v H)} = \frac{dK}{\Delta v(u_1 - c + \Delta u K)}$$

**equation involving infinitesimal quantities**

$$\frac{u_1 - c}{\Delta u} + K = A\left(\frac{v_1}{\Delta v} + H\right), \quad A = \text{const.}$$

$H$ and $K \to 0$ **as** $\xi \to -\infty$**, we get** $A = \dfrac{u_1 - c}{v_1} \dfrac{\Delta v}{\Delta u}$

# Mathematical Analysis (cont'd)

$H$ and $K \to 1$ **as** $\xi \to \infty$ **gives** $c = u_1 - \frac{\Delta u}{\Delta v} v_1$ **shock speed and** $H = K$**. These are HF's associated with specific volume and velocity.**

**The momentum eq. can be integrated to give**

$$H(\xi) = \frac{3\Delta p}{4\mu\Delta u} \int_0^{\xi} \exp\left[\frac{3\Delta p}{4\mu\Delta u}(\xi' - \xi)\right] L(\xi') d\xi$$

**This is a functional relation between** $H$ **and** $L$ **(associated with pressure).**

# Mathematical Analysis (contin'd))

**The energy eq. is:**

$$\alpha\beta v\Delta u L' + (1+\alpha)p\Delta v(L-H) - \Delta p\Delta v(L-H)^2 = 0$$

**Using the functional relation between $H$ and $L$, we get a nonlinear 2nd order ODE for $H$ that can be integrated to give:**

$$\frac{\beta\Delta u}{p_1}\left(1+\frac{\Delta v}{v_1}H\right)H' + \left(\frac{\Delta p}{p_1}+\frac{\gamma\Delta v}{v_1}\right)H + \frac{\gamma+1}{2}\frac{\Delta p}{p_1}\frac{\Delta v}{v_1}H^2 = 0$$

**If we reverse the role of $H$ and $\xi$, this equation is separable as follows.**

# Mathematical Analysis (cont’d)

$$\int_0^H \frac{\left(1 + \frac{\Delta v}{v_1} y\right)}{\frac{\gamma + 1}{2}\frac{\Delta p \Delta v}{p_1 v_1} y^2 + \left(\frac{\Delta p}{p_1} + \frac{\gamma \Delta v}{v_1}\right) y} dy = -\frac{p_1}{\beta \Delta u} \xi$$

**This equation is integrated using Mathematica 4. The functional relation between $H$ and $L$ can be integrated analytically using the energy relation to give:**

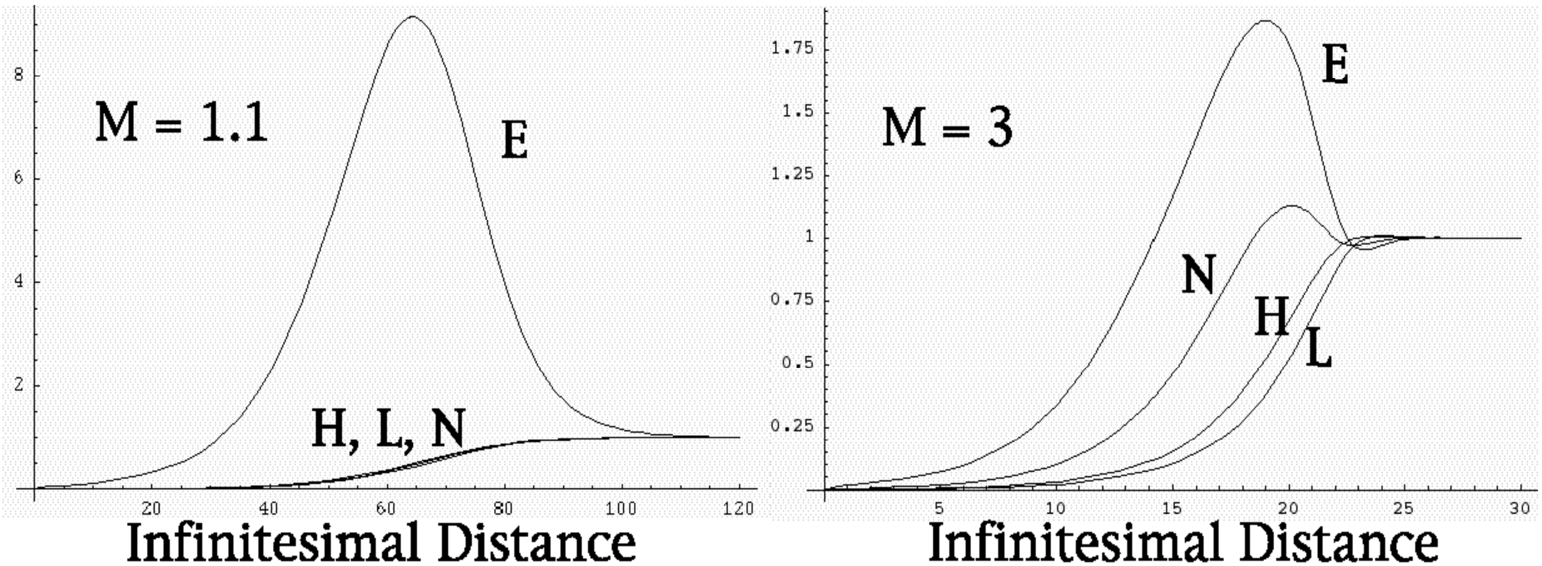
$$L = -p_1 \Delta v \left(\gamma + \frac{\gamma - 1}{2}\frac{\Delta p}{p_1} H\right) \Big/ \left[ v_1 \Delta p \left(1 + \frac{\Delta v}{v_1} H\right) \right]$$

# Results

To integrate the relation for $H$ we divide the length scale by the infinitesimal $\varepsilon$ so that we can visualize the results over reals. Once $H$ is obtained, $L$ is known in closed form. We have shown that $K = H$.

The HF's $E$ for entropy and $T$ for temperature are found from thermodynamic relations which depend on $H$ and $L$ (these are HF's for specific volume and pressure).

# Mathematica Calculations




**Constant viscosity and heat conductivity, Prandtl number = 3/4**

**Definitions of HF's: H specific volume (= K velocity), L Pressure, E entropy, N temperature**

# Concluding Remarks

– We have defined a class of HF's which differ in their microstructure on an infinitesimal interval in the halo of zero

– The derivatives of all orders of these HF's are defined and have a value in hyperreals over the infinitesimal region

– The product of each HF with another HF or the derivatives of a HF is defined unambiguously

– These functions all behave like Schwartz distributions on the test function spaces of Schwartz, i.e., they retain their macroscopic properties

– We have given an example of the application of these new class of HF's and their derivatives to shock structure problem obtaining well-known results